ABOUT THE MIXED ANDRÉ-OORT CONJECTURE: REDUCTION TO A LOWER BOUND FOR THE PURE CASE

ZIYANG GAO

Abstract. We prove that the mixed André-Oort conjecture holds for any mixed Shimura variety if a lower bound for the size of Galois orbits of special points in pure Shimura varieties is true. This generalizes the current results for mixed Shimura varieties of abelian type.

Résumé. Nous démontrons que la conjecture d'André-Oort pour toutes les variétés de Shimura mixtes, sous une borne inférieure pour la taille des orbites galoisiennes des points spéciaux. Ceci généralise les résultats connus pour les variétés de Shimura mixtes de type abélien.

1. Introduction

The goal of this note is to clarify the situation of the study to the mixed André-Oort conjecture following the Pila-Zannier method developped in [11] (see also [17] for a more detailed survey). The main result is to remove the assumption "of abelian type" in the known results.

Conjecture 1.1 (Mixed André-Oort)**.** *Let S be a mixed Shimura variety and let Σ be a set of special points in S. Then every irreducible component of the Zariski closure of Σ in S is a special subvariety.*

Before proceeding, we introduce some notations: let $(P, \mathcal{X})$ be a mixed Shimura datum and let K be a compact open subgroup of $P(\mathbb{A}_f)$. Then the mixed Shimura variety $M_K(P, \mathcal{X})$ is defined as

$$M_K(P, \mathcal{X}) := P(\mathbb{Q})\backslash\mathcal{X} \times P(\mathbb{A}_f)/K.$$

A **special subvariety** of $M_K(P, \mathcal{X})$ is then defined to be an irreducible component of $\overline{\mathcal{Y} \times \{p\}}$ (the image of $\mathcal{Y} \times \{p\}$ in $M_K(P, \mathcal{X})$), where $(Q, \mathcal{Y})$ is a mixed Shimura subdatum of $(P, \mathcal{X})$.

There is a number field $E(P, \mathcal{X})$ canonically associated to $(P, \mathcal{X})$, which is called the *reflex field* of $(P, \mathcal{X})$. Let $\pi\colon (P, \mathcal{X}) \to (G, \mathcal{X}_G)$ be the natural projection to the pure part of $(P, \mathcal{X})$. Then $E(P, \mathcal{X}) = E(G, \mathcal{X}_G)$. Let $K_G = \pi(K)$, then K_G is an open compact subgroup of $G(\mathbb{A}_f)$. Denote by

$$\mathrm{Sh}_{K_G}(G, \mathcal{X}_G) := G(\mathbb{Q})\backslash\mathcal{X}_G \times G(\mathbb{A}_f)/K_G$$

and $[\pi]\colon M_K(P, \mathcal{X}) \to \mathrm{Sh}_{K_G}(G, \mathcal{X}_G)$. The varieties $M_K(P, \mathcal{X})$, $\mathrm{Sh}_{K_G}(G, \mathcal{X}_G)$ and the map $[\pi]$ are all defined over $E(P, \mathcal{X}) = E(G, \mathcal{X}_G)$.

Let $x \in \mathcal{X}_G$ be any pre-special point, we shall denote by

(1) $T := \mathrm{MT}(x)$ which is a torus;
(2) $K_T := K_G \cap T(\mathbb{A}_f)$;
(3) K_T^m the maximal compact subgroup of $T(\mathbb{A}_f)$ containing K_T;

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(4) $i(T) := |\{p|\ K_{T,p} \neq K_{T,p}^m\}|$;
(5) L_x the splitting field of T;
(6) $\mathrm{Cl}(T) := T(\mathbb{Q})\backslash T(\mathbb{A}_f)/T(\widehat{\mathbb{Z}})$.

The main result is the following:

Theorem 1.2. *Let $M_K(P,\mathcal{X})$ be a mixed Shimura variety associated with the mixed Shimura datum $(P,\mathcal{X})$.*

Assume that there exist positive constants c_5, c_6, c_7 and c_8 (depending only on $(G,\mathcal{X}_G)$ and K_G, the numbering in conformity with [2, Theorem 1.2]*) such that for each pre-special point $x \in \mathcal{X}_G$, its image $[x,1]$ in the Shimura variety $\mathrm{Sh}_{K_G}(G,\mathcal{X}_G)$ satisfies (Conjecture 1.3 below or)*

$$|\operatorname{Gal}(\overline{\mathbb{Q}}/E(G,\mathcal{X}_G))\cdot[x,1]| \geqslant c_5 c_6^{i(T)}[K_T^m : K_T]^{c_7}|\operatorname{disc} L_x|^{c_8}.$$

Then the mixed André-Oort conjecture (Conjecture 1.1) holds for $M_K(P,\mathcal{X})$.

In order to prove the conjectural lower bound in Theorem 1.2, it suffices to prove the following conjecture, which is purely about class groups of tori:

Conjecture 1.3. *Let $\mathrm{Sh}_{K_G}(G,\mathcal{X}_G)$ be a pure Shimura variety and let $x \in \mathcal{X}_G$ be a pre-special point. Then there exist positive constants c and δ depending only on $\mathrm{Sh}_{K_G}(G,\mathcal{X}_G)$ such that*

$$|\operatorname{im}(\widetilde{r_T}\colon \mathrm{Cl}(L_x) \to \mathrm{Cl}(T))| \geqslant c D_T^{\delta}$$

where $\widetilde{r_T}$ is induced by the reciprocity morphism $r_T\colon \operatorname{Res}_{L_x/\mathbb{Q}}\mathbb{G}_{m,L_x} \to T$ and D_T is the quasi-discriminant of T (see [13, Section 7.1 and 3.5] *or* [16, Section 4.2 and 2.1.3] *for definitions).*

The fact that Conjecture 1.3 implies the conjectural lower bound in Theorem 1.2 is proved by Ullmo-Yafaev [16, Proposition 5.1] (also by Tsimerman [13, Theorem 7.1, Lemma 7.2] for the case $\mathcal{A}_g$). Also note that it suffices to prove Conjecture 1.3 for $\mathcal{A}_g$ and for pure Shimura varieties of adjoint type by the proof of Theorem 1.2 in §2. The conjectural lower bound is then proved unconditionally for $\mathcal{A}_g$ ($g \leqslant 6$) by Tsimerman [13] and under GRH for all pure Shimura varieties by Ullmo-Yafaev [16]. So we get

Theorem 1.4. *The mixed André-Oort conjecture (Conjecture 1.1) holds under GRH for all mixed Shimura varieties, and it holds unconditionally for any mixed Shimura variety whose pure part is a subvariety of $\mathcal{A}_6^N$ for any positive integer N.*

The proof is a combination of previous work of Pila-Tsimerman [9, 10], Klingler-Ullmo-Yafaev [6], Ullmo [14], Gao [5] and the recent result of Daw-Orr [2]. The difference between Theorem 1.2 (and Theorem 1.4) and the previous result of Gao [5, Theorem 13.6] is that we can remove the "of Abelian type" assumption thanks to the recent work of Daw-Orr [2].[1]

Achknowledgement. I would like to thank Emmanuel Ullmo for having asked me how to remove the "of abelian type" in the mixed case. I would like to thank Christopher Daw and Martin Orr for their explanation on their result, especially Section 4 of their paper. I would like to thank Jacob Tsimerman for pointing out to me that the conjectural lower bound can be reduced to the purely number theoretical question posed in Conjecture 1.3 and Yves André for

[1]For the traditional proof (initiated by Edixhoven [3]) of Conjecture 1.1 for pure Shimura varieties assuming GRH by Klingler-Ullmo-Yafaev [15, 7] (simplified by Daw [1]), there is no generalization to the mixed case to my knowledge.

suggesting me to change the previous title of this note to a more appropriate one. I would like to thank the referees for their suggestions to improving the note.

2. Proof of the main result

Let us prove Theorem 1.2. We use the notation of Conjecture 1.1 and Theorem 1.2.

Conjecture 1.1 is equivalent if we replace K by a neat subgroup. So we may assume that K is neat. Let Y be an irreducible component of $\overline{\Sigma}^{\mathrm{Zar}}$. Let $\mathcal{X}^+$ be a connected component of $\mathcal{X}$. Remark that

$$M_K(P,\mathcal{X}) = \coprod_{p\in P(\mathbb{Q})_+\backslash P(\mathbb{A}_f)/K} \Gamma_p\backslash\mathcal{X}^+,$$

where $\Gamma_p = P(\mathbb{Q})_+ \cap pKp^{-1}$, and $\Gamma_p\backslash\mathcal{X}^+ = \overline{\mathcal{X}^+\times\{p\}}$ in $M_K(P,\mathcal{X})$. Since Y is irreducible, there exists a $p\in P(\mathbb{A}_f)$ such that $Y\subset\overline{\mathcal{X}^+\times\{p\}}$. The goal is to prove that there exists a mixed Shimura subdatum $(Q,\mathcal{Y})$ of $(P,\mathcal{X})$ such that $Y=\overline{\mathcal{Y}^+\times\{p\}}$, where $\mathcal{Y}^+$ is a connected component of $\mathcal{Y}$. Hence we may and do assume that $p=1$.

Denote for simplicity Γ_1 by Γ and $S=\Gamma\backslash\mathcal{X}^+$. Denote by $\mathrm{unif}\colon \mathcal{X}^+\to S$. Then S is a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P,\mathcal{X}^+)$.

Replacing $(P,\mathcal{X}^+)$ be the smallest connected mixed Shimura subdatum $(Q,\mathcal{Y}^+)$ such that $\mathrm{unif}(\mathcal{Y}^+)$ contains Y and replacing S by $S_Q := \mathrm{unif}(\mathcal{Y}^+)$ does not affect the correctness of Conjecture 1.1, so we do these replacements. Now by [5, Theorem 12.2], there exists a normal subgroup $N\lhd P$ such that for the diagram

$$\begin{tikzcd}
(P,\mathcal{X}^+) \arrow[r, "\rho"] \arrow[d, "\mathrm{unif}"'] & (P',\mathcal{X}'^+) := (P,\mathcal{X}^+)/N \arrow[d, "\mathrm{unif}'"'] \\
S \arrow[r, "{[\rho]}"] & S'
\end{tikzcd}$$

- the union of positive-dimensional weakly special subvarieties of S' which are contained in $Y' := \overline{[\rho]Y}$ is NOT Zariski dense in Y';
- $Y=[\rho]^{-1}(Y')$.

The second bullet point tells us that Y is special iff Y' is special. By replacing (S,Y) by (S',Y'), we may assume that the union of positive dimensional weakly special subvarieties of S which are contained in Y is not Zariski dense in Y. So it suffices to prove the following claim:

Claim 1. *The set of special points in Y which do not lie in any positive dimensional special subvariety is finite (recall that we are under the assumption that the union of positive dimensional weakly special subvarieties of S which are contained in Y is not Zariski dense in Y).*

Now let us do some reductions:

(1) Replace P by $\mathrm{MT}(\mathcal{X}^+)$;
(2) By the Reduction Lemma [12, 2.26] and replacing Γ by a neat subgroup if necessary, we may furthermore assume that

$$(P,\mathcal{X}^+) = (G_0,\mathcal{D}^+)\times\prod_{i=1}^{r}(P_{2g},\mathcal{X}_{2g}^+)$$

where $(G_0,\mathcal{D}^+)$ is a pure Shimura datum, and that

$$\Gamma=\Gamma_0\times\prod_{i=1}^{r}\{\gamma\in P_{2g}(\mathbb{Z})|\gamma\equiv(0,1)\bmod N\}$$

for some $N>3$ even.

(3) Replace $(G_0,\mathcal{D}^+)$ by its adjoint $(G_0^{\mathrm{ad}},\mathcal{D}^+)$;

(4) Finally replace $(P,\mathcal{X}^+)$ by the smallest $(Q,\mathcal{Y}^+)$ such that $Y\subset\mathrm{unif}(\mathcal{Y}^+)$.

Remark that Y is defined over a number field, which we call k, because it contains a Zariski dense subset of special points. Then $E(G,\mathcal{X}_G)=E(P,\mathcal{X})\subset k$.

Now $\mathcal{X}^+\hookrightarrow\mathcal{D}^+\times\prod_{i=1}^{r}\mathcal{X}_{2g}^+$. So by [5, Proposition 4.3], we can identify $\mathcal{X}^+$ as a subspace of $\mathbb{C}^r\times\mathbb{R}^{2gr}\times(\mathcal{D}^+\times\mathbb{H}_g^{+r})$. Then any pre-special point is contained in

$$\mathbb{Q}^r\times\mathbb{Q}^{2gr}\times\left(\mathcal{D}^+\times(\mathbb{H}_g^+\cap M_{2g}(\overline{\mathbb{Q}}))^r\right).$$

Now let $\mathcal{F}$ be a fundamental set in $\mathcal{X}^+$ for the action of Γ on $\mathcal{X}^+$ as in [5, Section 10.1].

For any special point $s\in S$, take a representative $\widetilde{s}\in\mathcal{X}^+$ in $\mathcal{F}$. Write

$$\widetilde{s}=(\widetilde{s}_U,\widetilde{s}_V,x_0,x_1,...,x_r)$$

as the coordinates for $\mathbb{Q}^r\times\mathbb{Q}^{2gr}\times\left(\mathcal{D}^+\times(\mathbb{H}_g^+\cap M_{2g}(\overline{\mathbb{Q}}))^r\right)$. Then by choice of $\mathcal{F}$,

$$H(\widetilde{s}_U),H(\widetilde{s}_V)\ll N(s),\tag{2.1}$$

where $N(s)$ is the smallest positive integer q such that $q\widetilde{s}_U\in\mathbb{Z}^r$ and $q\widetilde{s}_V\in\mathbb{Z}^{2gr}$. Write for simplicity $x=(x_0,x_1,...,x_r)\in\mathcal{X}_G^+\subset\mathcal{D}^+\times(\mathbb{H}_g^+\cap M_{2g}(\overline{\mathbb{Q}}))^r$. Use the notations of the Introduction (above Theorem 1.2), we have that any $T_i:=\mathrm{MT}(x_i)$ is a quotient of T (for all $i=0,1,...,r$). So in particular,

- $[K_{T_i}^m:K_{T_i}]\leqslant[K_T^m:K_T]$;
- $i(T_i)\leqslant i(T)$;
- $L_{x_i}\subset L_x$.

Now by Daw-Orr [2, Theorem 1.4] (for $i=0$) and Pila-Tsimerman [9, Theorem 3.1] (together with Tsimerman [13, Lemma 7.2] for $i=1,...,r$), for any positive real number c_2, there exist positive constants c_1, c_3, c_4 (depending only on $\mathrm{Sh}_{K_G}(G,\mathcal{X}_G)$ and c_2) such that

$$H(x)=\max(H(x_0),H(x_1),...,H(x_r))\leqslant c_1c_2^{i(T)}[K_T^m:K_T]^{c_3}|\operatorname{disc}L_x|^{c_4}.\tag{2.2}$$

On the other hand for the special point $s\in S$ and for any $\varepsilon>0$, there exists $c(\varepsilon)>0$ such that (recall that $E(G,\mathcal{X}_G)=E(P,\mathcal{X})\subset k$)

$$\begin{aligned}|\operatorname{Gal}(\overline{\mathbb{Q}}/k)s|&\geqslant c(\varepsilon)N(s)^{1-\varepsilon}|\operatorname{Gal}(\overline{\mathbb{Q}}/k)\cdot[\pi]s| &&\text{by Gao [5, Theorem 13.3]}\\&=c(\varepsilon)N(s)^{1-\varepsilon}|\operatorname{Gal}(\overline{\mathbb{Q}}/k)\cdot[x,1]|\\&\geqslant c(\varepsilon)c_5N(s)^{1-\varepsilon}c_6^{i(T)}[K_T^m:K_T]^{c_7}|\operatorname{disc}L_x|^{c_8}&&\text{by assumption.}\end{aligned}\tag{2.3}$$

Now (2.1), (2.2) and (2.3) assert that there exists positive constants δ_0 and δ_1 depending only on $M_K(P,\mathcal{X})$ such that (remark that we can take in (2.2) $c_2=c_6$)

$$|\operatorname{Gal}(\overline{\mathbb{Q}}/k)s|\geqslant\delta_0H(\widetilde{s})^{\delta_1}.$$

Hence for $H(\widetilde{s})\gg0$, Pila-Wilkie [8, 3.2] implies that there exists $\sigma\in\operatorname{Gal}(\overline{\mathbb{Q}}/k)$ such that $\widetilde{\sigma(s)}$ is contained in a positive dimensional connected semi-algebraic subset $\widetilde{Z}$ of $\mathcal{X}^+$ contained in $\mathrm{unif}^{-1}(Y) \cap \mathcal{F}$. Let Z' be an irreducible component of $\overline{\mathrm{unif}(\widetilde{Z})}^{\mathrm{Zar}}$ containing $\sigma(s)$. Now the mixed Ax-Lindemann theorem (here we use the form [4, Theorem 3.7]) claims that Z' is weakly special, which is then special since it contains a special point $\sigma(s)$. But $\dim Z' > 0$ since $\dim \widetilde{Z} > 0$. Hence $\sigma^{-1}(Z')$ is special of positive dimension. To sum it up, for any special point $s \in Y$ with $H(\widetilde{s}) \gg 0$, s is contained in a positive dimensional special subvariety. Therefore the heights of the points in

$$\{\widetilde{s} \in \mathrm{unif}^{-1}(Y) \cap \mathcal{F} \text{ special and } \mathrm{unif}(\widetilde{s}) \text{ not contained in any positive dimensional special subvariety}\}$$

is uniformly bounded, and hence this set is finite by Northcott's theorem. Therefore we have proved Claim 1, and so Theorem 1.2.

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Institut des Hautes Études Scientifiques, Le Bois-Marie 35, route de Chartres, 91440 Bures-sur-Yvette, France

E-mail address: ziyang.gao@math.u-psud.fr